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Bohm's Interpretation of Quantum Mechanics and the Reconstruction of the Probability Distribution

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Abstract

Within Bohm's interpretation of quantum mechanics particles follow "classical" trajectories that are determined by the full solution of the time dependent Schrödinger equation. If this interpretation is consistent it must be possible to determine the probability distribution at time t, $\rho(x,t)$, from the probability distribution at time $t=0$, $\rho(x,0)$, by using these trajectories. In this paper it is shown that this is the case indeed.

1 Introduction

In 1952 David Bohm [1] proposed a new interpretation of quantum mechanics connecting "classical" trajectories to particles with a space-time probability distribution given by the full solution of the time dependent Schrödinger equation. A detailed presentation of this interpretation may be found for instance in [2], whereas its historical contingency is discussed in [3].

The main points of his interpretation, which are of concern here, may be described as follows (for simplicity the space dimension is taken to be 1 and units $\hbar = m = 1$ are used). Let $\psi(x,t)$ be the solution of the time dependent Schrödinger equation for a certain system, that is

$$i\frac{\partial}{\partial t}\psi(x,t) = H\psi(x,t) \tag{1}$$

where H denotes the Hamilton operator of the system. Then one decomposes $\psi(x,t)$ in its absolute value and the corresponding phase which results in

$$\psi(x,t) = R(x,t)\exp\left[iS(x,t)\right] \tag{2}$$

The phase $S(x,t)$ determines the "classical" motion by requiring that the momentum of a particle at position x at time t is given by

$$p = \frac{dx}{dt} = \frac{\partial S(x,t)}{\partial x} \tag{3}$$

Solving (3) for a particle with initial position x_0 at time t_0 attributes to this particle a classical trajectory $x(x_0,t)$ subject to the initial condition $x_0 = x(x_0,t_0)$. Obviously one may use these trajectories to transport the probability distribution

$$\rho(x,t_0) = \left|\psi(x,t_0)\right|^2 = R(x,t_0)^2 \tag{4}$$

in time by moving the initial distribution at points x_0 to the points $x(x_0,t)$, which is suggested by Bohm's interpretation of quantum mechanics. Of course this transport in time cannot be the complete procedure since the normalization condition of the probability distribution must be maintained too. In the next section it will be shown how one can reconstruct the probability distribution $\rho(x,t)$ from the initial one, $\rho(x,t_0)$, by using classical trajectories, the solutions of (3), and that the result agrees with the usual quantum mechanical one, namely

$$\rho(x,t) = \left|\psi(x,t)\right|^2 \tag{5}$$

2 Reconstruction of the Probability Distribution

Let the probability distribution at time $t = 0$ be (t_0 is chosen to be 0 for simplicity)

$$\widetilde{\rho}(x,0) = \int \delta(x - x_0) f(x_0) dx_0 = f(x) \tag{6}$$

This is the continuous analogue of particles distributed at points x_0 with some probability $f(x_0)$. According to Bohm's interpretation the points x_0 move to the points $x(x_0,t)$ in time and therefore following this idea the probability distribution at time t must be given by

$$\widetilde{\rho}(x,t) = \int \delta((x - x(x_0,t)) f(x_0) dx_0 \tag{7}$$

where $x(x_0,t)$ is the solution of (3) with the appropriate initial condition. From (7) it is easy to obtain the wanted result because the integration can be done without difficulty by means of the δ-function. The zero of the δ-function is given by $x_0(x,t)$, the inverse function to $x(x_0,t)$. This gives, using differentiation rules for inverse functions,

$$\widetilde{\rho}(x,t) = \left| \frac{\partial x_0(x,t)}{\partial x} \right| f(x_0(x,t)) \tag{8}$$

and the final result is

$$\widetilde{\rho}(x,t) = \left| \frac{\partial x_0(x,t)}{\partial x} \right| \widetilde{\rho}\left(x_0\left(x,t\right),0\right) \tag{9}$$

That this result, obtained by using Bohm's interpretation of quantum mechanics, agrees with the usual quantum mechanical result (5) can be shown as follows. Together with the current density

$$j(x,t) = \int \frac{\partial x(x_0,t)}{\partial t} \delta(x - x(x_0,t)) f(x_0,t) dx_0 \tag{10}$$

$\widetilde{\rho}(x,t)$ as defined in (7) fulfills the continuity equation

$$\frac{\partial \widetilde{\rho}(x,t)}{\partial t} + \frac{\partial j(x,t)}{\partial x} = 0 \tag{11}$$

But $j(x,t)$, due to the momentum definition (3), can also be written in the form

$$j(x,t) = \int \frac{\partial S(x,t)}{\partial x} \delta(x - x(x_0,t)) f(x_0,t) dx_0 = \frac{\partial S(x,t)}{\partial x} \widetilde{\rho}(x,t) \tag{12}$$

Thus $\widetilde{\rho}(x,t)$ fulfills the same continuity equation as $\rho(x,t)$ defined in (5). At time $t=0$ both agree, if $f(x)$ is chosen to be

$$f(x) = |\psi(x,0)|^2 \tag{13}$$

The uniqueness of the solution of the continuity equation implies then that

$$\widetilde{\rho}(x,t) \equiv \rho(x,t) \quad if \quad \widetilde{\rho}(x,0) = \rho(x,0) \tag{14}$$

This shows that the reconstruction of the probability distribution at time $t=0$ following Bohm's ideas agrees with the usual quantum mechanical result and the explicit expression is given by (9). But the imagination that the probability distribution at time t ist just obtained by the flow of particles from the initial distribution following the trajectories $x(x_0,t)$ is in general wrong due to the factor $\left|\frac{\partial x_0(x,t)}{\partial x}\right|$ in (9). This factor could also be interpreted as a normalization factor since due to it the normalization of the wave function is valid for all times

$$1 = \int |\psi(x_0,0)|^2 \, dx_0 = \int |\psi(x_0(x,t),0)|^2 \left|\frac{dx_0(x,t)}{dx}\right| dx \tag{15}$$

The next section will show the role of this factor within three different examples explicitly.

3 Examples

3.1 The Coherent State of the Harmonic Oscillator

The normalized coherent state with amplitude d is given by ($\omega = 1$)

$$\psi_d(x,t) = \frac{1}{\pi^{\frac{1}{4}}} \exp\left\{-i\left[\frac{t}{2} - \frac{d^2}{4}\sin 2t + dx\sin t\right] - \frac{1}{2}\left[x - d\cos t\right]^2\right\} \tag{16}$$

Which gives for the absolute value $R(x,t)$ and the phase factor $S(x,t)$

$$R(x,t) = \frac{1}{\pi^{\frac{1}{4}}} \exp\left[-\frac{1}{2}(x - d\cos t)^2\right] \tag{17}$$

$$S(x,t) = -\frac{t}{2} + \frac{d^2}{4}\sin 2t - dx\sin t \tag{18}$$

For the definition of the momentum one obtains therefore

$$p = \frac{\partial S}{\partial x} = -d\sin t \tag{19}$$

resulting in the equation of motion

$$\frac{dx}{dt} = -d\sin t \tag{20}$$

with the solution

$$x(x_0,t) = d\,(\cos t - 1) + x_0 \tag{21}$$

From this the inverse function is given by

$$x_0(x,t) = x - d\,(\cos t - 1) \tag{22}$$

and therefore

$$\left|\frac{\partial x_0(x,t)}{\partial x}\right| = 1 \tag{23}$$

as expected since a coherent state does not change its shape in time. Obviously the reconstructed probability distribution agrees with the quantum mechanical result as it has to be

$$\begin{aligned} \rho(x,0) &= R(x,0)^2 = \frac{1}{\pi^{\frac{1}{2}}}\exp\left[-\left(x-d\right)^2\right] \Longrightarrow \\ \rho(x_0(x,t),0) &= \frac{1}{\pi^{\frac{1}{2}}}\exp\left\{-\left[x_0(x,t)-d\right]^2\right\} = R(x,t)^2 \end{aligned} \tag{24}$$

3.2 The Wave Packet for a Moving Free Particle

The normalized wave function for a free particle with a Gaussian distribution around $x = 0$ at time $t = 0$ and for which the peak of the probability distribution moves with constant velocity (here taken to be one) in time is given by

$$\psi(x,t) = \frac{1}{\pi^{\frac{1}{4}}\sqrt{1+it}}\exp\left[-\frac{x^2-2ix+it}{2\,(1+it)}\right] \tag{25}$$

From this the absolute value and the phase of the wave function are given by

$$R(x,t) = \frac{1}{\sqrt[4]{(t^2+1)\,\pi}}\exp\left[-\frac{1}{2}\frac{\left(t-x\right)^2}{t^2+1}\right] \tag{26}$$

$$S(x,t) = \frac{1}{4i}\ln\frac{1-it}{1+it} + \frac{tx^2+2x-t}{2(1+t^2)} \tag{27}$$

with the corresponding "classical" momentum

$$p = \frac{\partial S}{\partial x} = \frac{tx+1}{1+t^2} \tag{28}$$

The equation of motion

$$\frac{dx}{dt} = \frac{tx+1}{1+t^2} \tag{29}$$

has the solution

$$x = t + x_0\sqrt{1+t^2} \tag{30}$$

whose inverse is given by

$$x_0 = \frac{x-t}{\sqrt{1+t^2}} \tag{31}$$

and

$$\left|\frac{\partial x_0(x,t)}{\partial x}\right| = \frac{1}{\sqrt{1+t^2}} \tag{32}$$

which is just the right factor to correct the normalization for the broadening of the wave packet in time. Again one obtains complete agreement with Bohm's interpretation

$$\begin{aligned} \rho(x,0) &= R(x,0)^2 = \frac{1}{\sqrt{\pi}}\exp(-x^2) \Longrightarrow \left|\frac{\partial x_0(x,t)}{\partial x}\right| \rho(x_0(x,t),0) \\ &= \frac{1}{\sqrt{(1+t^2)\pi}}\exp\left[-\frac{(x-t)^2}{1+t^2}\right] = R(x,t)^2 \end{aligned} \tag{33}$$

One may be tempted to interpret the factor $\left|\frac{\partial x_0(x,t)}{\partial x}\right|$ to be responsible only for the normalization of the wave function without any relation to its shape. That this imagination is completely wrong will be shown in the last example.

3.3 The Harmonic Oscillator Again

Instead of a coherent state a superposition of the ground state with the first excited state of the harmonic oscillator is considered. The time dependent solution for this case is given by

$$\psi(x,t) = \sqrt{\frac{2}{3}}\frac{1}{\pi^{1/4}}\exp\left(-\frac{x^2+it}{2}\right)\left[1+x\exp\left(-it\right)\right] \tag{34}$$

and therefore

$$R(x,t) = \sqrt{\frac{2}{3}}\frac{1}{\pi^{1/4}}\exp\left(-\frac{x^2}{2}\right)\sqrt{1+2x\cos t+x^2} \tag{35}$$

$$S(x,t) = -\frac{t}{2}+\frac{1}{2i}\ln\frac{1+x\exp\left(-it\right)}{1+x\exp(it)} \tag{36}$$

Proceeding in the usual way one has therefore to solve the differential equation

$$\frac{dx}{dt} = -\frac{\sin t}{1+2x\cos t+x^2} \tag{37}$$

which has the implicit solution

$$\begin{aligned} &\frac{3}{4}\sqrt{\pi}\mathrm{erf}\left(x\right)-\exp\left(-x^2\right)\cos t-\frac{1}{2}x\exp\left(-x^2\right) \\ =\ &\frac{3}{4}\sqrt{\pi}\mathrm{erf}\left(x_0\right)-\exp\left(-x_0^2\right)\cos t-\frac{1}{2}x_0\exp\left(-x_0^2\right) \end{aligned} \tag{38}$$

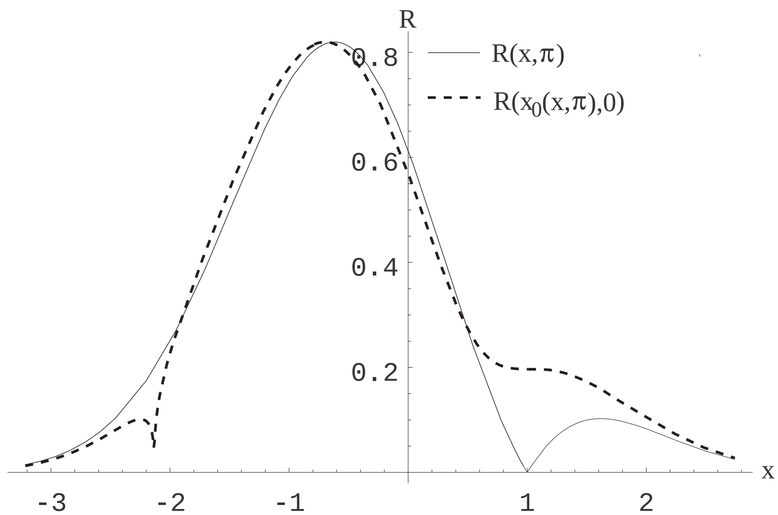

Figure 1: Comparison of the exact result $R(x,\pi)$ and the result obtained by using "classical" trajectories but without the "normalization" factor

One can numerically extract from this implicit solution $x(x_0,t)$ or the inverse $x_0(x,t)$ and compare the exact result $\rho(x,t)$ with the result obtained from using "classical" trajectories but without the factor $\left|\frac{\partial x_0(x,t)}{\partial x}\right|$, that is with $\rho(x_0(x,t),0)$ only. This is shown in Fig. 1 for the time $t=\pi$ (for simplicity the square roots are plotted, that is $R(x,t)$ and $R(x_0(x,t),0)$). The disagreement in shape is obvious, but full agreement is achieved if the factor $\left|\frac{\partial x_0(x,t)}{\partial x}\right|$ is taken into account.

4 Conclusion

It has been shown that Bohm's interpretation of quantum mechanics gives the correct answer for the probability distribution at later times when applied to the probability distribution of a system at time $t=t_0$. Of course no new results can be obtained in this way, since first one has to know the wave function of the system for all times. But the main point is that the interpretation in itself does not lead to any contradictions if one takes it literally. Nevertheless one

has to be very cautious in applying this interpretation. As is shown by the last example of the superposition of two harmonic oscillator eigenstates, one cannot view the particle flow as just transferring the initial proabability distribution to a final one, which has to be normalized only, but agrees in shape with the result obtained in this way. On the contrary, the factor $\left|\frac{\partial x_0(x,t)}{\partial x}\right|$ may change this shape quite drastically in general and makes it very difficult to visualize what is going on.

References

[1] D. Bohm, Phys. Rev. 85 (1952) 166, 180.

[2] D. Bohm, B.J. Hiley, "The Undivided Universe: An Ontological Interpretation of Quantum Theory", Routledge, London, 1993.

[3] J.T. Cushing, "Quantum Mechanics: Historical Contingency and the Copenhagen Hegemony", University of Chicago Press, Chicago, 1994